

Convective Lyapunov Spectra

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Abstract. - We generalize the concept of convective (or velocity-dependent) Lyapunov exponent $\Lambda(v)$ to an entire spectrum $\Lambda(v, n)$. Our results are supported by the consistency between the outcome of the chronotopic approach [S. Lepri et al. *J. Stat. Phys.*, 82 5/6 (1996) 1429] and a more direct method. There exists a critical integrated density $n = n_c$, beyond which the convective exponent exhibits a discontinuous dependence on the velocity, which originates from the appearance of multiple branches. This phenomenon can be traced back to a change of concavity of the so-called *temporal* Lyapunov spectrum for $n > n_c$, which is therefore a dynamical invariant.

Introduction. – The linear-stability analysis of an N dimensional chaotic dynamics typically amounts to computing N Lyapunov exponents (LEs) $\{\lambda_l\}$ with $l = 1, \dots, N$ (typically ordered from the largest to the most negative one). In a spatially extended system, the Lyapunov spectrum depends only on the integrated density $n = l/L$ ($\lambda(n)$) if the system size L is large enough [1]. Since this tool does not provide any information on the spatial propagation of perturbations, a new indicator was introduced in Ref. [2], the convective (comoving) Lyapunov exponent (CLE) $\Lambda(v)$, which quantifies the maximal growth rate of an initially localized perturbation, in a frame that moves with a velocity v . Later, it was recognized that the CLE can be derived from a general theory, based on the so-called *chronotopic* approach [3], which deals with the wider class of perturbations with a spatially exponential profile $\exp(\mu i)$, where i denotes a discrete spatial variable and μ is a free parameter. As a result, one can, e.g., define the generalized *temporal* Lyapunov $\lambda(n, \mu)$ spectrum. The CLE $\Lambda(v)$ can be thereof computed as a Legendre transform of the maximal temporal Lyapunov exponent, i.e. $\lambda(0, \mu)$ [4].

In systems with left-right spatial symmetry, $\Lambda(v)$ is symmetric around $v = 0$, where it attains its maximum value which coincides with the standard maximum LE. The largest velocity $v = v_c$ such that $\Lambda(v) \geq 0$ is the max-

imal propagation velocity of infinitesimal perturbations. In convectively unstable systems $\Lambda(0) < 0$ and the (positive) maximum convective exponent is attained for some nonzero velocity. In fact, a somehow similar approach was developed by Huerre and Monkewitz [5] to characterize absolute and convective instabilities in open flows and more recently extended by Sandstede and Scheel to deal with generic boundary conditions [6].

In this Letter we go beyond the computation of the growth rate of the local amplitude of a perturbation, turning our attention to the evolution of volumes of generic dimension. The chronotopic approach offers a straightforward way to determine an entire spectrum of *convective* LEs, by extending the notion of Legendre transform from the maximum exponent (i.e. $n = 0$) to generic values of the integrated density n . However, it would be desirable to give a more direct definition as well. In principle, the most appropriate starting point for a direct definition of a spectrum of CLE is the approach developed in [7], where an ensemble of linearly independent initial conditions (localized within a window of size L) was freely let evolve to thereby determine volume expansion rates within the very same window. As a result, it was noticed that meaningful generalized Lyapunov exponents could be defined by simultaneously letting the time T and the size L tend to infinity, with constant ratio $g = T/L$. The standard

Lyapunov spectrum is recovered for $g \rightarrow 0$, while in the opposite limit $g \rightarrow \infty$, all exponents coincide with the maximum. The dependence on g expresses the fact that the growth along different directions is affected in a different manner by the local expansion and by diffusion. In principle, this approach can be implemented for moving windows, too; however, the unavoidable dependence of the additional parameter g , reduces the appeal of such a direct method. Accordingly, we have preferred to complement the moving-window approach with some boundary conditions to get rid of the difficulty of dealing with an open system. In practice, this is the setup proposed in Ref. [8].

Altogether, in this Letter we compare this latter method with the chronotopic approach, finding that they are mutually consistent. Moreover, the chronotopic approach is by and large the most accurate and this has allowed discovering serious numerical difficulties that easily lead to artifacts in the direct computation of the convective Lyapunov spectra (even in the simple case of coupled maps, herein investigated). Moreover, the chronotopic approach has revealed that the lower part of the convective spectrum is characterized by the existence of multiple solutions. The phenomenon appears for $n > n_c$ and is associated to a change of concavity in the temporal Lyapunov spectrum. The critical density n_c is, by construction, a dynamically invariant dimension density (analogous to the dimension density of the stable manifold or to the Kaplan-Yorke dimension density), but the physical meaning of n_c still needs to be clarified.

Altogether the consistency between the two methods confirms the conjecture that the chronotopic approach “encodes” all stability properties of one-dimensional spatio-temporal systems, that are eventually contained in the corresponding entropy potential [3].

Theory. – First, we introduce the proper formalism with reference to a standard model of coupled maps,

$$y_{t+1}^i = f \left[(1 - \varepsilon) y_t^i + \frac{\varepsilon}{2} (y_t^{i-1} + y_t^{i+1}) \right], \quad (1)$$

where $i = 1, \dots, N$ and t are the spatial and temporal (integer) indices, respectively, while $\varepsilon \in (0, 1)$ represents the diffusive coupling and $f(y)$ is a map of the unit interval onto itself. We start by investigating the evolution of an infinitesimal perturbation δy_t^i , initially localized in a finite region of length $M + 1 \ll N$ centered around the origin ($\delta y_0^i = \xi^i \Theta(M/2 - i) \Theta(i + M/2)$, where Θ is the Heaviside function and the ξ^i 's are iid random variables). The standard convective Lyapunov exponent is defined as

$$\Lambda(v, 0) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{|\delta y_t^{vt}|}{|\delta y_0^0|} \quad (2)$$

where the “0” in parentheses signifies that we refer to the maximum exponent. The formula can be easily extended to generic dynamical systems, by replacing the absolute value $|\delta y_t^i|$ with any norm quantifying the amplitude of

the perturbation on the site i at time t . The question addressed in this Letter is the extension of this definition to quantify not only the growth rate of the amplitude but also the expansion/contractions along the additional directions. In order to address this issue, it is necessary to explore the spatial structure of the perturbations. In the standard approach, it is sufficient to first let the perturbation evolve freely in the space and, afterwards, to focus the attention on specific world lines ($i = vt$). Willing to go beyond the maximal convective LE, one should consider a set of localized but linearly independent initial conditions. Accordingly, a theoretical and practical difficulty emerges: that of defining a proper orthogonalization strategy to avoid the convergence towards the same direction. In the case of the standard Lyapunov spectrum, this problem is solved by resorting to the Gram-Schmidt technique [9]. However, here it is not obvious how to do that. The natural idea of referring to the space covered by the perturbations at a given time leads to the unavoidable conclusion that one would measure only the tip of the spectrum, since we would have to rescale by a growing coefficient. Therefore, one is led to restrict the orthogonalization to a moving window of fixed size $L < M + 1$, so that the computation of convective Lyapunov spectra for different velocities would require performing separate simulations. This idea, which was already suggested in [8], is not the optimal solution, as the initial perturbations are not let to freely evolve, but artificial modifications have to be introduced on the boundaries to confine the evolution within the prescribed window. Nevertheless, since it turns out that boundary conditions do not matter for large enough L (see below), we can at least claim that the approach is meaningful.

Let us now be more specific and explain how the LE can be determined in a frame that moves with a generic velocity v in a discrete spatio-temporal lattice. We introduce two types of “moves” in tangent space: the first corresponds to a static window (**0**); the second to a right-shift by a single site (**1**, without loss of generality we discuss only windows moving to the right). The corresponding rules are,

$$\delta y_{t+1}^k = m_t^i \left[\frac{\varepsilon}{2} \delta y_t^{j-1} + (1 - \varepsilon) \delta y_t^j + \frac{\varepsilon}{2} \delta y_t^{j+1} \right] \quad (3)$$

where $1 \leq k \leq L$, i denotes the absolute position in the lattice and $m_t^i = f'[(1 - \varepsilon)y_t^i + \varepsilon(y_t^{i-1} + y_t^{i+1})/2]$. **0**- and **1**-iterations correspond to $j = k$ and $j = k + 1$, respectively. The restriction of the rule to a finite interval requires extra assumptions for δy_t^{L+1} and δy_t^0 (δy_t^{L+2}), to close the model for a **0** (**1**) move. We have typically worked by assuming all of them to be equal to zero, but we have initially verified that the same results are obtained also for different choices (e.g., $\delta y_t^0 = \delta y_t^1$). As a last detail, it is necessary to specify the sequence of **0**s and **1**s that is used to study the velocity v . Given that v is obviously equal to the fraction of **1**s, we have typically selected (simple) rational numbers, choosing the most uniform sequence with

a fixed density (e.g., for $v = 2/5$, **010100101010001...**). As a result, for any given velocity v , we can determine the whole spectrum $\Lambda(v, n)$, where $n = l/L$ and l means we refer to the l th largest exponent.

An alternative approach consists in considering the evolution of a perturbation with an exponential profile, namely $\delta y_t^i = \Phi_t^i e^{\mu i}$. This requires studying the evolution equation

$$\Phi_{t+1}^i = m_t^i \left[\frac{\varepsilon}{2} e^{-\mu} \Phi_t^{i-1} + (1 - \varepsilon) \Phi_t^i + \frac{\varepsilon}{2} e^{-\mu} \Phi_t^{i+1} \right], \quad (4)$$

where $i = 1, \dots, N$ and periodic boundary conditions for Φ_t^i can be safely assumed. By solving the model (4) for different values of μ , one obtains the temporal spectrum $\lambda(n, \mu)$.

We claim that the convective Lyapunov spectrum can be obtained by a Legendre transform of $\lambda(n, \mu)$,

$$\Lambda(v, n) = \lambda(n, \mu) - \mu v \quad (5)$$

where $v = d\lambda/d\mu$ (and the set of transformations is completed by the symmetric expression $\mu = d\Lambda/dv$). Altogether Eq. (5) generalizes to $n > 0$ the transformation that has been proved for $n = 0$ [4]. Its original justification is based on the observation that the profile of an initially localized perturbation is, at time t , locally exponential with a decay rate μ that depends on the position $i = vt$. For $n > 0$, it is not clear which profile one should refer to. For this reason the above definition is essentially a formal one.

Numerical Results. – We have tested both approaches, by studying a chain of coupled logistic maps, i.e. with $f(y) = ry(1 - y)$ ($r = 4$), $y \in [0; 1]$ and $\varepsilon = 1/3$ (these are the same parameter values adopted in Ref. [8]).

In Fig. 1 we plot the temporal spectra for $N = 100$ and some values of n (namely, $n = 0, 0.5, 0.75, 0.9$ and 1). We have also verified that N is large enough to ensure the thermodynamic limit. The uppermost curve corresponds to the maximum exponent that is known to grow monotonously for increasing (in absolute value) μ . On the other hand, the lowermost curve, which corresponds to the minimum LE not only is non monotonous, but even exhibits a singular behavior for $|\mu| \approx 1.31$.

Convective exponents vs velocity. In Fig. 2a we plot the convective spectra obtained by Legendre transforming two of the curves reported in Fig. 1. The upper solid curve is the standard convective spectrum. It starts from a maximum value for $v = 0$ that corresponds to the usual maximum LE and crosses the zero axis at $v_c = 0.510(5)$ which indicates the maximal propagation velocity of perturbations. The lower solid curve corresponds to $n = 0.5$ and the very fact it is nearly zero for $v = 0$ indicates that approximately half of the standard LEs are positive.

In order to check the equivalence between the direct and the chronotopic approach, we have computed the convective spectra also by iterating localized perturbations,

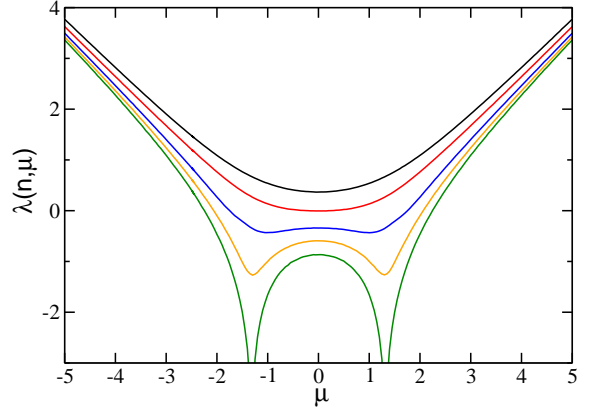


Fig. 1: (Color Online) Temporal Lyapunov exponents $\lambda(n, \mu)$ versus μ for a chain of $N = 100$ coupled logistic maps with $r = 4$ and $\varepsilon = 1/3$. From top to bottom the curves refer to $n = 0, 0.5, 0.75, 0.9$ and 1 .

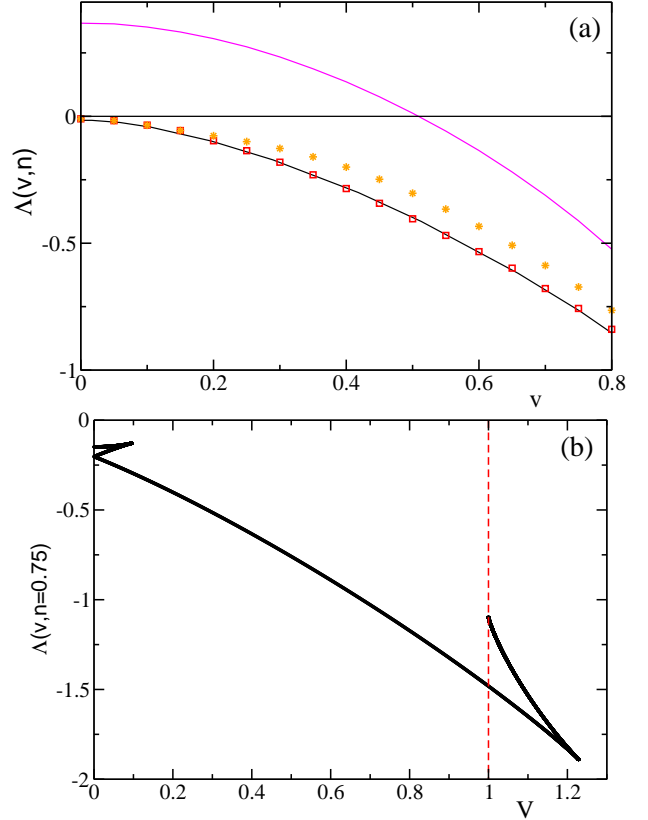


Fig. 2: (Color Online) Convective exponents $\Lambda(v, n)$ vs. the velocity of the comoving reference frame for Logistic (a) and Bernoulli (b) coupled map lattices. (a) The convective exponents have been obtained for $n = 0$ (upper curve) and $n = 0.5$ (lower curve). The solid lines have been obtained by Legendre transforming (see Eq. 5) the curves reported in the previous figure. The symbols correspond to direct estimates of the convective exponents in double precision for $n = 0.5$ and $L = 100$: (orange) asterisks refer to $\gamma = 0$, while (red) squares to $\gamma = 1.0$. (b) The reported convective spectrum has been analytically determined by performing the Legendre transform of Eq. (7) for $n = 0.75$ and $N = 100$.

implementing the method described in the previous section. The (orange) asterisks in Fig. 2a correspond to the outcome of simulations performed with $L = 100$. The agreement is rather poor and does even get worse upon increasing the window size L (not reported). However, we discovered that the problem is not a conceptual, but a numerical one. The reason is that the Lyapunov vectors in the moving windows are exponentially localized around the left border of the window. Accordingly, many of their components are so small that double-precision accuracy (i.e. with 14-15 digits) is not sufficient, especially for large values of L . As a matter of fact we have verified that by employing extended precision (i.e. with approximately 30 digits) the agreement increases. However, an even better agreement can be obtained already with double precision, by using weighted Euclidean. More precisely, given any two vectors $\mathbf{u} = \{u^i\}$, $\mathbf{v} = \{v^i\}$, we define the scalar product as

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i e^{\gamma i} \quad (6)$$

where γ is a free parameter. Formally speaking, it is well known that Lyapunov exponents are independent of the norm that is chosen, i.e., in this case, of γ . Nonetheless, the numerical accuracy may significantly depend on γ . In fact, by comparing double with extended precision for different values of γ we concluded that $\gamma = 1$ is a nearly optimal choice¹. The data reported in Fig. 2a (red squares) indeed confirm the increased agreement with the chronotopic results. However, a proper selection of γ does not solve completely the problem: for large velocities and larger values of L , numerical accuracy remains a serious issue.

Anyway, the relevant message that comes from the numerical analysis is that the definition of convective Lyapunov spectra through the chronotopic approach is not just formal but provides the correct answer and, moreover, the method is far more reliable than the direct one, since one does not face accuracy problems. Incidentally, it is the comparison between the two approaches that has allowed discovering the serious problems of numerical accuracy that affect the direct approach and thereby the simulations reported in Ref. [8]. Last but not least, for large values of the integrated density n , the non-monotonicity of $\lambda(n, \mu)$ leads to the emergence of extra branches in the convective spectrum, as shown in Fig. 2b (which refers to Bernoulli maps $f(x) = 2x \bmod(1)$ with the same coupling constant as before). The origin of the three branches at small velocities will be addressed in the following section, while the coexistence of branches at large velocities is totally irrelevant. In fact, it arises in an unphysical region, since $v > 1$ cannot be obtained in a lattice with nearest neighbor coupling. We interpret the very existence of the two branches as an instance of a phase velocity that can be faster than “light velocity” without causing any paradox.

¹At least for not too large velocities, otherwise the choice of the norm does not help enough

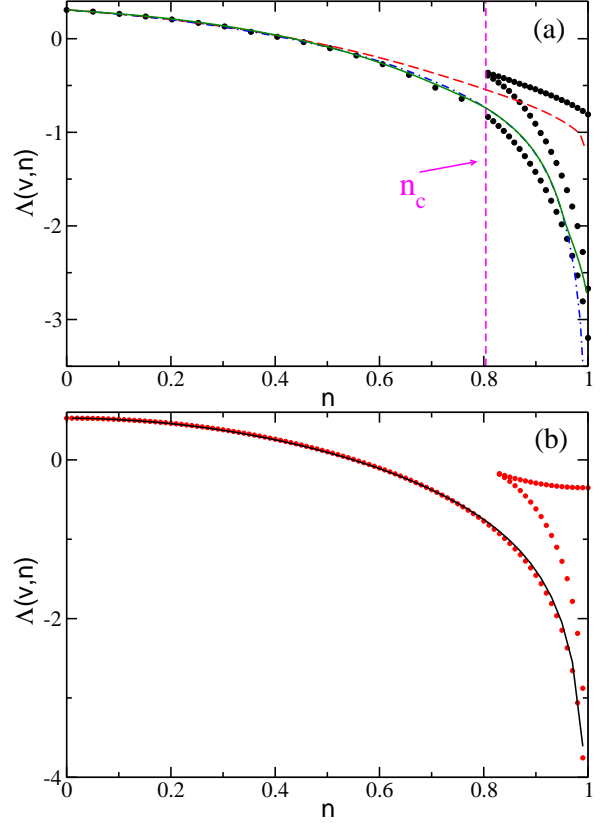


Fig. 3: (Color Online) Spectrum of the convective exponents $\Lambda(v, n)$. (a) Coupled logistic maps for $v = 1/5$. The solid lines correspond to direct measurements. From top to bottom: $L = 100$ and $\gamma = 0$ (red dashed line); $L = 200$ and $\gamma = 2$ (blue dot-dashed line); $L = 100$ and $\gamma = 2$ (green solid line). Full circles correspond to the Legendre transform for $N = 100$. Finally, the vertical dot-dashed (magenta) line indicates the n_c -value. (b) Bernoulli maps for $v = 1/3$, estimated analytically via the Legendre transform (red circles) and directly by diagonalizing a constant 50×50 matrix (solid line) with extended precision.

Convective exponents vs density. We now compare the convective spectra obtained for a given velocity, as it helps clarifying the behavior for large integrated densities. An example is reported in Fig. 3a for $v = 1/5$. Above a certain density n_c the spectrum obtained via the Legendre transform displays three different branches: the lower one is associated to positive μ -values, while the other two to negative μ . Once again the standard direct estimates suffer problems of numerical accuracy (see the upper solid curve in Fig. 3a). The improved simulations (for $\gamma = 2$) seem to converge towards the lower branch, but there is still some discrepancy and it is difficult to decide whether this is due to finite-size corrections.

In order to obtain a more convincing evidence, we now consider a model of Bernoulli maps. Since the multipliers are constant in space and time, we expect smaller finite size corrections and, moreover, the estimation of the convective spectra is simpler as it reduces to the diagonalization of a matrix (for rational velocities). In fact,

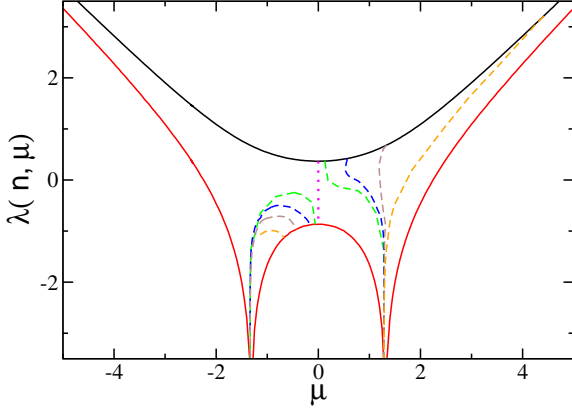


Fig. 4: (Color Online) Isolines in the plane (λ, μ) corresponding to a constant velocity v for coupled logistic maps. The vertical dotted line refer to $v = 0$, the other dashed lines from left to right for $\mu > 0$ (resp. right to left for $\mu < 0$) correspond to $v = 0.05, 0.20, 0.50$ and 0.95 with $N = 100$.

in this model, the tangent matrices (see Eq. (3)) depend only whether the move is of type **0** or **1**. Therefore, given a rational velocity, characterized by a periodic sequence of **0**s or **1**s (with period P), it is sufficient to multiply P of such matrices and thereby determine the eigenvalues. The results for $v = 1/3 = (001001001\dots)$ are reported in Fig. 3b, where one can appreciate that the agreement between the lower branch and the direct method is already quite impressive for a window of size $L = 50$.

In order to shed some light on the origin of these branches, we have plotted the paths in (λ, μ) -plane that correspond to spectra with different velocities (see Fig. 4). They are the isolines where the slope $d\lambda/d\mu = v$ stays constant. The vertical line at $\mu = 0$ is the path for the standard Lyapunov spectrum. However, for finite velocities the line breaks into two components that lie in the positive and negative μ half-planes, respectively: the former one corresponds to the lower branch in Fig. 3, while the latter one gives rise to the two upper branches. Altogether it is reasonable to conjecture that the reason for discarding the upper branches is that they correspond to negative μ values, i.e. to profiles that are larger on the right side. In fact, this is impossible unless one assumes the presence of some source of “noise” on the right of the moving window.

A moment’s reflection suggests that the mathematical origin of two branches in Fig. 4 is the change of concavity of $\lambda(n, \mu)$ (as a function of μ) upon increasing n : it suddenly enforces points with positive derivative to jump from the right to the left side of $\mu = 0$. Once again the Bernoulli maps allow for an accurate investigation.

The expression for the temporal exponent is [3],

$$\lambda(n, \mu) = \log a + \frac{1}{2} \log |(1 - \varepsilon)^2 + 2\varepsilon(1 - \varepsilon) \cosh \mu \cos \pi n + \varepsilon^2 (\cosh^2 \mu - \sin^2 \pi n)| \quad (7)$$

By expanding this expression for small values of μ , it is

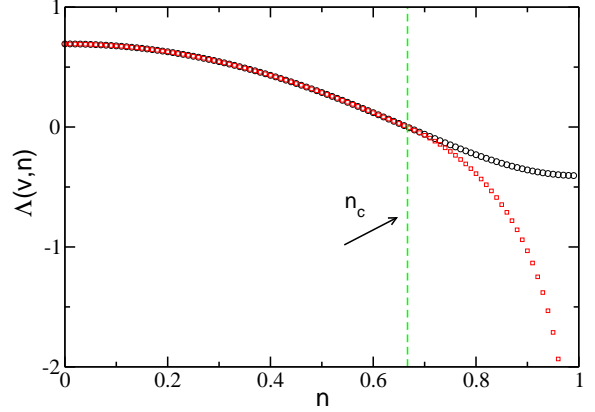


Fig. 5: (Color Online) Spectrum of the convective exponents $\Lambda(v, n)$ for coupled Bernoulli maps for $v = 0$ (circles) and $v = 0.001$ (squares), estimated by Legendre transforming of Eq. (7). The parameters are $a = 2$ and $\varepsilon = 1/3$.

easy to verify that the quadratic term in μ changes sign for

$$n = n_c \equiv 1 - \frac{\arccos[\varepsilon/(1 - \varepsilon)]}{\pi}, \quad (8)$$

provided that $\varepsilon < 1/2$. As a matter of fact we have estimated the spectrum for the convective exponents for $v = 0$ and $v = 0.001$. In Fig. 5. one can appreciate that for $n < n_c$ they are nearly identical, while for $n > n_c$, they separate out by a finite amount as a result of the selection of the lower branch. Now, we can return to $\Lambda(v, n)$ as a function of v for fixed n (illustrated in Fig. 2 for $n = 0.5$) and conclude that $\Lambda(v, n)$ must have a discontinuity in $v = 0$, if $n > n_c$.

Discussion and conclusions. — In this Letter, we have shown that the notion of convective Lyapunov exponent can be extended from the maximum to an entire spectrum. The comparison between two different methods reveals a rather complex scenario with several issues that need be further clarified. An example is the dimension density n_c which separates a standard behavior from the appearance of multiple branches. Its very existence is connected to a change of concavity in the temporal Lyapunov spectra. We have verified that in coupled maps it generally exists for not-too-large coupling ($\varepsilon < 1/2$ in Bernoulli maps) and preliminary simulations confirm the existence of such a critical density also in Stuart-Landau oscillators [10]. Nevertheless, its physical meaning is rather unclear. We can only claim that the dimension n_c is a dynamical invariant (as it follows from a general property of the Lyapunov spectrum) like the density of unstable directions (fraction of positive Lyapunov exponents), the Kaplan-Yorke dimension and the dimension of physical modes [11]. Another question concerns the upper branches that we have dismissed as irrelevant, but could play some role in specific physical contexts.

Finally, the direct definition is still partially unsatisfactory, as it involves the addition of in-principle-unnecessary

boundary conditions. An open system approach such as that developed in [7] would be much more appealing, but it requires incorporating the additional parameter $g = T/L$ in the current theory.

An alternative idea could be that of referring to covariant Lyapunov vectors [12]: if one could indeed “build” the initial perturbation by using only such vectors, one would be automatically assured that no more-unstable degrees of freedom are going to be excited. The problem is that one first needs to evolve back and forwards the chain on sufficiently long time scales to allow the perturbation propagate over sufficiently large distances to measure asymptotic quantities.

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